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DESIGN TECHNIQUES OF DIGITAL PROPORTIONAL-PLUS-INTEGRAL and PROPORTIONAL-PLUS-INTEGRAL-PLUS-DERIVATIVE CONTROLLERS FOR LINEAR DISCRETE-TIME MULTIVARIABLE SYSTEMS

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by

Ahmed I.A. Salama*

ABSTRACT

Time-domain design techniques of digital proportional-plus-integral (PI) and proportional-plus-integral-plus-derivative (PID) controllers for linear discrete-time multivariable systems are presented. The proposed design approach is based on the null-space formulation of the eigenstructure problem. Two cases are considered where the plant state or output is available for measurement. In the case of state feedback the resulting controller will assign a prescribed set of self-conjugate eigenvalues and a corresponding set of admissible self-conjugate generalized eigenvectors to the closed-loop system matrix. However, for output feedback a two-stage design procedure is adopted, where we first determine state feedback matrices which assign a prescribed set of self-conjugate eignevalues and a corresponding set of admissible self-conjugate generalized eigenvectors to a particular closed-loop system matrix, then those state feedback gain matrices are used to determine the feedback gain matrices of the output PID controller. The proposed techniques are flexible and straightforward compared with known techniques.

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TECHNIQUES DE CONCEPTION DE REGULATEURS NUMERIQUES PROPORTIONNELS/PAR INTÉGRATION ET PROPORTIONNELS/PAR INTÉGRATION/PAR DÉRIVATION POUR SYSTÈMES MULTIVARIABLES TEMPORELS, DISCRETS ET LINÉAIRES

par

Ahmed I.A. Salama*

RÉSUMÉ

Des techniques de conception à domaine temps de régulateurs numériques proportionnels/par intégration (PI) et proportionnels/par intégration/par dérivation (PID) pour systèmes multivariables temporels, discrets et linéaires, sont présentées. La conception proposée se fonde sur la formulation dans l'espace zéro du problème aux valeurs propres. Deux cas sont considérés : quand l'état du système peut être mesuré, et quand sa sortie peut l'être. Dans la réaction d'état, le régulateur assigne un ensemble prescrit de valeurs propres autoconjuguées et un ensemble correspondant de vecteurs propres généralisés autoconjugués admissibles à la matrice système à boucle fermée. Dans la réaction de sortie, on adopte une conception à deux étapes. On détermine d'abord les matrices de réaction d'état qui assignent un ensemble prescrit de valeurs propres autoconjuguées et un ensemble correspondant de vecteurs propres généralisés autoconjugués admissibles à une matrice système à boucle fermée, puis ces matrices de gain de réaction d'état

servent à déterminer les matrices de gain de réaction de la sortie du régulateur PID. Les techniques proposées sont souples et se comparent directement aux techniques connues.

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INTRODUCTION

The goal of any industrial process control is to operate the process in an optimal fashion based on a specified performance criterion. This can be facilitated by keeping the different process variables (flowrate, pressure, temperature etc.) at desired values. These process variables are often controlled satisfactorily by means of PID controllers. Usually the measurement and control requirements of an application dictate the type of control electronics required i.e. whether they should be analogue or digital. In the past, most of the instrumentation and controllers were analogue, but today, most of the development of the computer technology and increase in computer utilization in process control has resulted in the utilization of digital instrumentation and controllers. Consequently, in direct digital control (DDC) and distributed microprocessor supervisory control (DMSC) the process variables are controlled by digital PID algorithms.

Recently, Seraji (1983) presented simple methods for the design of digital PI, proportional-plus-derivative (PD) and PID controllers for pole placement in linear discrete-time multivariable systems, where the system output is available for measurement. Some conditions were derived on the number of eigenvalues to be placed arbitrarily. Seraji's (1983) design techniques are in the frequency domain, and are mainly addressed to the eigenvalue assignment problem. However, in many practical situations, we are required to control the modes of the closed-loop system response, (i.e. modal control). Therefore the problem of eigenvalue-eigenvector assignment will be the subject of the present paper.

In this paper, a time-domain approach for designing digital PI and PID controllers for linear discrete-time multivariable systems is adopted. The null space formulation of the eigenstructure problem is utilized in determining the controller feedback gain matrices. In PART 1. and PART 2. of the paper the design of state PI and PID controllers are investigated, respectively. The resulting gain matrices assign a prescribed set of self-conjugate eigenvalues and a corresponding set of admissible self-conjugate generalized eigenvectors to the closed-loop system matrix. In PART 3. the design of output PID controller is investigated. A two-stage design procedure is proposed, where first we utilize the results of PART 2. to determine a set of state feedback gain matrices, then those gain matrices are used to determine the output feedback gain matrices of the output PID controller.

The proposed design techniques have the following advantages: i) they are in the time-domain, thus there is no computation of the transfer function matrix of the closed-loop system from its state variable formulation, which can be highly inaccurate (Bosley et al, 1972), ii) it imposes no structural constraint on the feedback gain matrices (i.e. unit rank), thus allowing greater freedom in the design. Moreover, the PI and PID controllers are superior to any other conventional proportional controller in rejecting any system disturbance due to the existing integral part. The design techniques presented in this contribution can be considered as an extension of the problem of eigenvalue-eigenvector assignment in linear continuous multivariable system using a PI controller considered by Salama and Gourishankar (1981).

PART 1. DIGITAL PI CONTROLLER DESIGN UTILIZING STATE FEEDBACK

PROBLEM STATEMENT

Consider the linear discrete-time, time-invariant multivariable system described in state space form as follows

$$X(k+1) = AX(k) + BU(k), \qquad (1)$$
$$k = 0,1,2,.....$$

where X(k) and U(k) are the state and input vectors and of dimensions n and m, respectively, at sample k, and A,B are constant matrices of appropriate dimensions. Let us introduce a q-dimensional integrator state vector $\mathit{X}(k)$ which satisfies

$$\mathit{X}(k+1) = \mathit{X}(k) + T(Q_I X(k) - Q_I X_d), \qquad (2)$$
$$k = 0,1,2,.....$$

where Q_I is a constant matrix of dimension qxn, T is the sampling interval and $Q_I X_d$ is a desired vector for $Q_I X(k)$. The state feedback law is assumed to be of the form (see Figure 1)

$$U(k) = G_p X(k) + G_I \mathit{X}(k), \qquad (3)$$

where G_p and G_I are constant matrices representing the proportional and integral feedback gain matrices of the PI controller and of dimensions mxn and mxq, respectively. Without loss of generality we can assume T is unity. Substitution of (3) in (1) yields the closed-loop system

$$X(k+1) = (A+BG_p)X(k) + BG_I \mathit{X}(k). \qquad (4)$$

Then (2) and (4) can be written in augmented state variable form as

$$Z_c(k+1) = A_c Z_c(k) + \Theta, \qquad (5)$$
$$k = 0,1,2,.....$$

where

$$A_c = \begin{bmatrix} A+BG_p & BG_I \\ Q_I & I_q \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 \\ -Q_I X_d \end{bmatrix}, \qquad (6)$$

$$Z_c = \begin{bmatrix} X(k) \\ \mathit{X}(k) \end{bmatrix} \qquad (7)$$

and I_q is the identity matrix of dimension q. The closed-loop matrix A_c has a dimension $\eta = n+q$. The study of the eigenvalue assignment problem requires the establishment of state controllability. To this end the closed-loop matrix A_c in (6) can be rewritten as

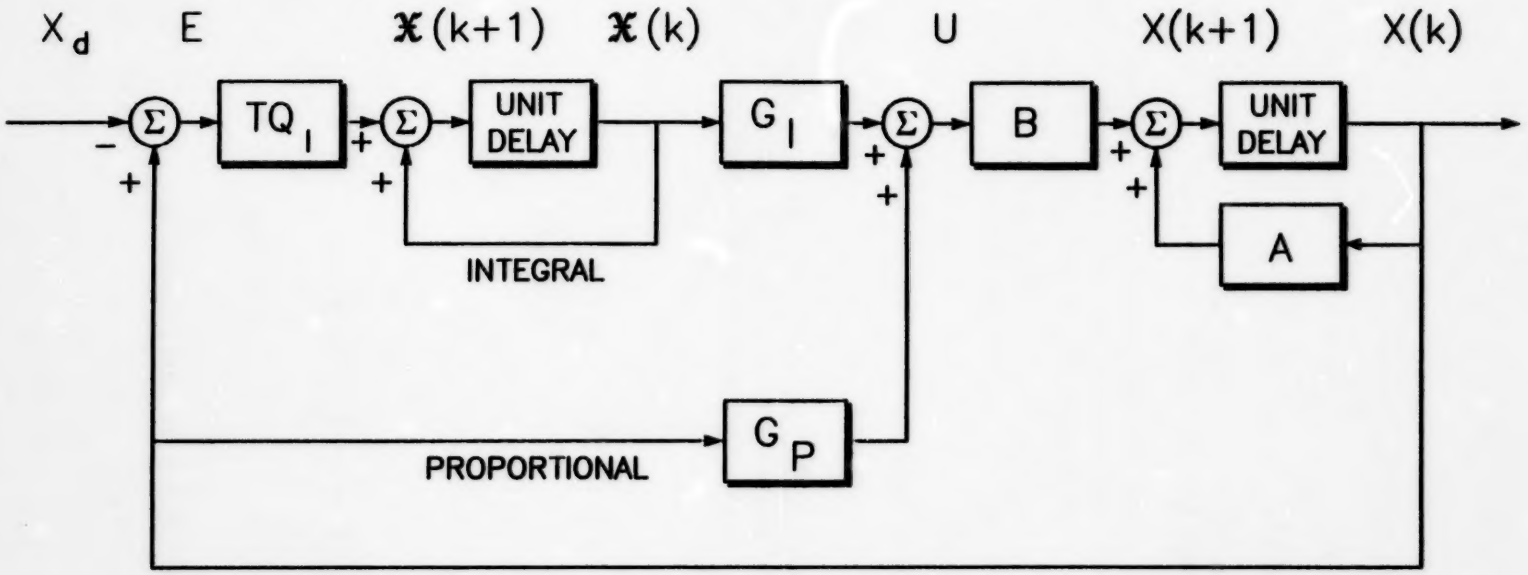

Figure 1

A Linear Discrete–Time Multivariable System With Digital PI Controller

$$A_c = \begin{bmatrix} A & 0 \\ Q_I & I_q \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [G_p + G_I] \cdot \tag{8}$$

The state controllability requires that the matrix:

$$S = \begin{bmatrix} B & AB & A^2B & \cdots\cdots\cdots & A^{\eta-1}B \\ 0 & Q_IB & Q_I(B+AB) & \cdots\cdots\cdots & Q_I \sum_{i=0}^{\eta-2} A^i B \end{bmatrix} , \tag{9}$$

has a rank η. If this rank condition is not met, adjustment of the number of inputs and/or number of integrator inputs or another choice of Q_I may correct this situation.

The problem of eigenvalue-generalized eigenvector assignment of the system represented by (1) via a proportional and integral feedback law of the form (3), where the integrator state is governed by (2) is one of determining the real matrices G_p and G_I such that the closed-loop system matrix A_c has a prescribed set of self-conjugate eigenvalues and a corresponding set of admissible self-conjugate generalized eigenvectors.

EIGENVALUE-GENERALIZED EIGENVECTOR ASSIGNMENT

Eigenvalue not Equal One

Consider a particular set of self-conjugate eigenvalues of A_c, $\lambda_i \neq 1$, $i=1,2,\ldots,\ell$, $\ell \leq \eta$. Let the eigenvalue λ_i has a geometric multiplicity k_i and algebraic multiplicity m_i. The generalized eigenvectors $U^{i,\mu,\rho}$'s associated with λ_i must satisfy

$$[A_c - \lambda_i I] U^{i,\mu,\rho} = U^{i,\mu,\rho-1} , \tag{10}$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu} , \qquad \mu = 1,2,\ldots\ldots,k_i, \qquad i = 1,2,\ldots\ldots,\ell,$$

with $U^{i,\mu,0} = 0$, also $m_{i,\mu}$ must satisfy

$$m_i = \sum_{\mu=1}^{k_i} m_{i,\mu} . \tag{11}$$

The dimension of A_c, namely η, is related to m_i as

$$\eta = \sum_{i=1}^{\ell} m_i . \tag{12}$$

Let the generalized eigenvector $U^{i,\mu,\rho}$ be partitioned as

$$U^{i,\mu,\rho} = ((U^{i,\mu,\rho}_1)', (U^{i,\mu,\rho})')' , \tag{13}$$

where the ' denotes vector transpose. The dimensions of the subvectors $U_1^{i,\mu,\rho}$ and $\mathcal{U}^{i,\mu,\rho}$ are n and q, respectively. Using (6) and (10) we get

$$((A-\lambda_i I)+BG_P)U_1^{i,\mu,\rho} + BG_I\mathcal{U}^{i,\mu,\rho} = U_1^{i,\mu,\rho-1}, \tag{14}$$

$$Q_I U_1^{i,\mu,\rho} + (1-\lambda_i)\mathcal{U}^{i,\mu,\rho} = \mathcal{U}^{i,\mu,\rho-1}. \tag{15}$$

From (15) we get

$$\mathcal{U}^{i,\mu,\rho} = (\mathcal{U}^{i,\mu,\rho-1} - Q_I U_1^{i,\mu,\rho})/(1-\lambda_i) . \tag{16}$$

From (16) it is obvious that $\mathcal{U}^{i,\mu,\rho}$ can be expressed in terms of $U_1^{i,\mu,\rho}$ Then (14) can be written in matrix form as

$$[A-\lambda_i I, B]\begin{bmatrix} U_1^{i,\mu,\rho} \\ W^{i,\mu,\rho} \end{bmatrix} = U_1^{i,\mu,\rho-1} , \tag{17}$$

$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \quad \mu = 1,2,\ldots\ldots,k_i,$
$i = 1,2,\ldots\ldots,\ell,$

where

$$W^{i,\mu,\rho} = G_P U_1^{i,\mu,\rho} + G_I\mathcal{U}^{i,\mu,\rho} . \tag{18}$$

In the case where the eigenvalues λ_i, $i=1,2,\ldots\ldots,\ell$, form a complete set of self-conjugate eigenvalues and the integers $m_{i,\mu}$ and k_i are chosen so that the entire set of vectors $U^{i,\mu,\rho}$'s not only satisfies (14,15), but also linearly independent and self-conjugate, then the real feedback gain matrices can be expressed using (17) and (18) as

$$[G_P, G_I]\begin{bmatrix} \ldots\ldots\ldots, & U_1^{i,\mu,\rho} & , \ldots\ldots\ldots \\ \ldots\ldots\ldots, & \mathcal{U}^{i,\mu,\rho} & , \ldots\ldots\ldots \end{bmatrix}$$

$$= [\ldots\ldots\ldots, W^{i,\mu,\rho} , \ldots\ldots\ldots] , \tag{19}$$

$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \quad \mu = 1,2,\ldots\ldots,k_i,$

$i = 1,2,\ldots\ldots,\ell,$

or in a compact form as

$$[G_P, G_I]\ U = W . \tag{20}$$

The choice of the subvectors $U^{i,\mu,\rho}$'s is decided by the control system designer to achieve desired system response or modes shaping. Consequently, by using the set of subvectors $U^{i,\mu,\rho}$'s, the system matrices A,B and (17) we can determine the set of vectors $W^{i,\mu,\rho}$'s and the matrices U and W can be formed. The matrix U is square with dimension n , if U is invertible then

$$[G_P, G_I] = WU^{-1} . \tag{21}$$

The determination of the feedback gain matrices G_P and G_I of the PI controller using (21) will complete the design procedure. In the previous analysis, it was assumed that the eigenvalues not equal one. However for the sake of presentation completeness the case where an eigenvalue is one will be discussed in the next section.

Eigenvalue $\lambda_i=1$

Let λ_i be unity eigenvalue, then (14,15) reduce to

$$((A-I)+BG_P)U_1^{i,\mu,\rho} + BG_I U^{i,\mu,\rho} = U_1^{i,\mu,\rho-1}, \tag{22}$$

$$Q_I U_1^{i,\mu,\rho} = U^{i,\mu,\rho-1}. \tag{23}$$

We can assume without loss of generality that the matrix Q_I be expressed as $[I_q,0]$. Then from (23) it is obvious that q elements of $U_1^{i,\mu,\rho}$ are known in terms of $U^{i,\mu,\rho-1}$. The remaining (n-q) elements of $U_1^{i,\mu,\rho}$ and $U^{i,\mu,\rho}$ are to be assigned. Equation (22) can be written in a matrix form as

$$[A-I,B]\begin{bmatrix} U_1^{i,\mu,\rho} \\ W^{i,\mu,\rho} \end{bmatrix} = U_1^{i,\mu,\rho-1}, \tag{24}$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i,$$

where

$$W^{i,\mu,\rho} = G_P U_1^{i,\mu,\rho} + G_I U^{i,\mu,\rho}. \tag{25}$$

Using (24) and (25), we can express the real feedback gain matrices as

$$[G_P,G_I]\begin{bmatrix} \ldots\ldots, & U_1^{i,\mu,\rho}, & \ldots\ldots \\ \ldots\ldots, & U^{i,\mu,\rho}, & \ldots\ldots \end{bmatrix} = [\ldots\ldots, \quad W^{i,\mu,\rho}, \ldots\ldots]. \tag{26}$$

For a general eigenvalue spectrum, we use the procedure discussed for nonunity eigenvalue and the present procedure for the unity eigenvalues, then employing (19,26,20,21), we can determine the PI controller gain matrices G_P and G_I. A numerical example will be considered in the next section to illustrate the proposed design technique .

ILLUSTRATIVE EXAMPLE

Let us consider a third order system of the form (1), where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and three choices of Q_I are treated:

$$\text{(a) } Q_I = [1\ 0\ 0], \quad \text{(b) } Q_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{(c) } Q_I = I_3.$$

Using (9), it can be shown that they have state controllability for cases (a) and (b).

The design objective is to determine the PI controller gain matrices G_P and G_I such that the closed-loop system matrix A_c has the eigenvalue spectrum whose description is:

$$\begin{aligned} &\lambda_1 = 0,\ k_1=2, \\ &\text{case (a)}\ \ m_1=4,\ \ m_{1,1}=m_{1,2}=2\ , \\ &\text{case (b)}\ \ m_1=5,\ \ m_{1,1}=2, m_{1,2}=3, \\ &\text{case (c)}\ \ m_1=6,\ \ m_{1,1}=m_{1,2}=3\ , \end{aligned}$$

and the corresponding set of admissible generalized eigenvector-subvectors $U_1^{i,\mu,}$'s from:

$$U_1^{1,1,1} = \begin{bmatrix} 1 \\ -1 \\ 0.5 \end{bmatrix}, \quad U_1^{1,1,2} = \begin{bmatrix} 1 \\ 0 \\ -0.75 \end{bmatrix}, \quad U_1^{1,1,3} = \begin{bmatrix} 0.25 \\ 0.75 \\ 0 \end{bmatrix},$$

$$U_1^{1,2,1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad U_1^{1,2,2} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}, \quad U_1^{1,2,3} = \begin{bmatrix} 1 \\ -1 \\ 0.75 \end{bmatrix}.$$

Since the eigenvalue spectrum is the zero eigenvalue, then the results obtained for zero eigenvalue can be applied. Using (17-21), the resulting feedback gain matrices of the PI controllers for cases (a),(b) and (c) are:

$$[G_P, G_I] = \begin{bmatrix} -2 & -1 & -2 & -1 \\ 1.25 & 0 & 0 & 0.75 \end{bmatrix},$$

$$[G_P, G_I] = \begin{bmatrix} -2 & -1 & -2 & -1 & 0 \\ 0.5 & -1 & 0 & 0.5 & 1 \end{bmatrix},$$

$[G_P, G_I]$ = do not exist,

respectively. It can be verified that the resulting gain matrices G_P and G_I in cases (a) and (b) assign the desired eigenvalue spectrum and the set of admissible generalized eigenvectors to the closed-loop system matrix A_c. In case (a), the Jordan canonical form of A_c is composed of two blocks each of dimension $m_{1,1}=m_{1,2}=2$, while in case (b) the Jordan canonical form of the A_c is composed of two blocks of dimensions $m_{1,1}=2$ and $m_{1,2}=3$, then it can be shown that the state $Q_I X(k)$ will reach the desired state $Q_I X_d$ in two and three samples for cases (a) and (b), respectively.

PART 2. DIGITAL PID CONTROLLER DESIGN UTILIZING STATE FEEDBACK

PROBLEM STATEMENT

Consider the linear discrete-time, time-invariant multivariable system described in state space form as follows

$$X(k+1) = AX(k) + BU(k), \qquad k = 0,1,2,..... \tag{27}$$

where X(k) and U(k) are the state and input vectors and of dimensions n and m, respectively, at sample k, and A,B are constant matrices of appropriate dimensions. Let us introduce a q-dimensional integrator state vector $X(k)$ which satisfies

$$X(k+1) = X(k) + T(Q_I X(k) - Q_I X_d), \qquad k = 0,1,2,..... \tag{28}$$

where Q_I is the matrix of dimension qxn and T is the sampling interval, $Q_I X_d$ is a desired vector for $Q_I X(k)$ and T is the sampling interval. The state feedback law is assumed to be of the form (see Figure 2)

$$U(k) = G_P X(k) + G_I X(k) + G_D(X(k) - X(k-1))/T, \tag{29}$$

where G_P, G_I and G_D are constant matrices representing the proportional, integral and derivative feedback gain matrices of the PID controller and of dimensions mxn, mxq and mxn, respectively. Without loss of generality we can assume T is unity. Substitution of (29) in (27) yields the closed-loop system

$$X(k+1) = (A+BG_P+BG_D)X(k) - BG_D X(k-1) + BG_I X(k). \tag{30}$$

Let us denote X(k-1) and X(k) by $Z_1(k)$ and $Z_2(k)$, respectively, then (28) and (30) can be written in state variable form as

$$Z_c(k+1) = A_c Z_c(k) + \Theta, \qquad k = 0,1,2,..... \tag{31}$$

where

$$A = \begin{bmatrix} 0 & I_n & 0 \\ -BG_D & A+BG_P+BG_D & BG_I \\ 0 & Q_I & I_q \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 \\ 0 \\ -Q_I X_d \end{bmatrix}, \tag{32}$$

$$Z_c = \begin{bmatrix} Z_1(k) \\ Z_2(k) \\ X(k) \end{bmatrix}. \tag{33}$$

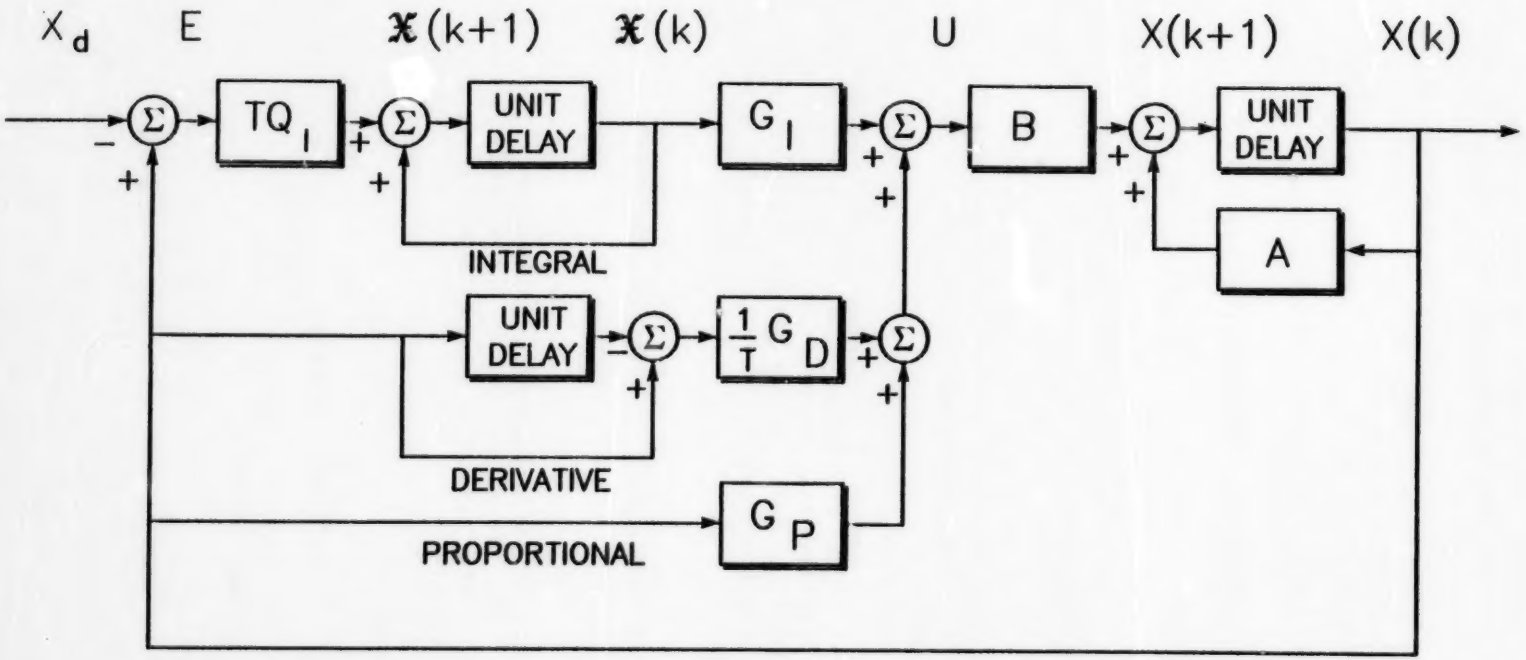

Figure 2
A Linear Discrete–Time Multivariable System with Digital PID Controller

The closed-loop matrix A_c has a dimension $\eta = 2n+q$. The study of the eigenvalue assignment problem requires the establishment of state controllability. To this end the closed-loop matrix A_c in (32) can be rewritten as

$$A_c = \begin{bmatrix} 0 & I_n & 0 \\ 0 & A & 0 \\ 0 & Q_I & I_q \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ 0 \end{bmatrix} [-G_D,\ G_P+G_D,\ G_I] \quad . \tag{34}$$

Using (34), the state controllability requires that the matrix

$$S = \begin{bmatrix} 0 & B & AB & \cdots\cdots\cdots & A^{\eta-2}B \\ B & AB & A^2B & \cdots\cdots\cdots & A^{\eta-1}B \\ 0 & Q_IB & Q_I(B+AB) & \cdots\cdots\cdots & Q_I \sum_{i=0}^{\eta-2} A^iB \end{bmatrix}, \tag{35}$$

has a rank η. If this rank condition is not met, adjustment of the number of inputs and/or number of integrator inputs or another choice of the matrix Q_I may correct this situation.

The problem of eigenvalue-generalized eigenvector assignment of the system represented by (27) via a proportional, integral and derivative feedback law of the form (29), where the integrator state is governed by (28) is one of determining the real matrices G_P, G_I and G_D such that the closed-loop system matrix A_c has a prescribed set of self-conjugate generalized eigenvalues and a corresponding set of admissible self-conjugate generalized eigenvectors.

EIGENVALUE-GENERALIZED EIGENVECTOR ASSIGNMENT

Eigenvalue Not Equal Zero or One

Consider a particular set of self-conjugate eigenvalues of A_c, $\lambda_i \neq 0$ or 1, $i=1,2,\ldots\ldots,\ell$, $\ell \leq \eta$. Let the eigenvalue λ_i has a geometric multiplicity k_i and algebraic multiplicity m_i. The generalized eigenvectors $U^{i,\mu,\rho}$'s associated with λ_i must satisfy

$$[A_c-\lambda_iI]U^{i,\mu,\rho} = U^{i,\mu,\rho-1},$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i, \qquad i = 1,2,\ldots\ldots,\ell, \tag{36}$$

with $U^{i,\mu,0} = 0$ also $m_{i,\mu}$ must satisfy

$$m_i = \sum_{\mu=1}^{k_i} m_{i,\mu} \ . \tag{37}$$

The dimension of A_c, namely η, is related to m_i as

$$\eta = \sum_{i=1}^{\ell} m_i .$$

Let the generalized eigenvector $U^{i,\mu,\rho}$ be partitioned as

$$U^{i,\mu,\rho} = ((U_1^{i,\mu,\rho})', (U_2^{i,\mu,\rho})', (\mathcal{U}^{i,\mu,\rho})')', \tag{39}$$

where the ' denotes vector transpose. The dimensions of the subvectors $U_1^{i,\mu,\rho}$, $U_2^{i,\mu,\rho}$ and $\mathcal{U}^{i,\mu,\rho}$ are n, n and q, respectively. Using (32) and (36) we get

$$-\lambda_i U_1^{i,\mu,\rho} + U_2^{i,\mu,\rho} = U_1^{i,\mu,\rho-1}, \tag{40}$$

$$-BG_D U_1^{i,\mu,\rho} + ((A-\lambda_i I)+B(G_P+G_D))U_2^{i,\mu,\rho} + BG_I \mathcal{U}^{i,\mu,\rho} = U_2^{i,\mu,\rho-1}, \tag{41}$$

$$Q_I U_2^{i,\mu,\rho} + (1-\lambda_i)\mathcal{U}^{i,\mu,\rho} = \mathcal{U}^{i,\mu,\rho-1}. \tag{42}$$

From (40) and (42) we get , respectively,

$$U_1^{i,\mu,\rho} = (U_2^{i,\mu,\rho} - U_1^{i,\mu,\rho-1})/\lambda_i, \tag{43}$$

$$\mathcal{U}^{i,\mu,\rho} = (\mathcal{U}^{i,\mu,\rho-1} - Q_I U_2^{i,\mu,\rho})/(1-\lambda_i) . \tag{44}$$

From (43) and (44) it is obvious that $U_1^{i,\mu,\rho}$ and $\mathcal{U}^{i,\mu,\rho}$ can be expressed in terms of $U_2^{i,\mu,\rho}$. Then (41) can be written in matrix form as

$$[A-\lambda_i I, B]\begin{bmatrix} U_2^{i,\mu,\rho} \\ W^{i,\mu,\rho} \end{bmatrix} = U_2^{i,\mu,\rho-1}, \tag{45}$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i,$$
$$i = 1,2,\ldots\ldots,\ell,$$

where

$$W^{i,\mu,\rho} = G_P U_2^{i,\mu,\rho} + G_I \mathcal{U}^{i,\mu,\rho} + G_D(U_2^{i,\mu,\rho} - U_1^{i,\mu,\rho}). \tag{46}$$

In the case where the eigenvalues λ_i, $i=1,2,\ldots\ldots,\ell$, form a complete set of self-conjugate eigenvalues and the integers $m_{i,\mu}$ and k_i are chosen so that the entire set of vectors $U^{i,\mu,\rho}$'s not only satisfies (40-42),

but also linearly independent and self-conjugate, then the real feedback gain matrices can be expressed using (45) and (46) as

$$[G_P,G_I,G_D]\begin{bmatrix} \cdots\cdots\cdots, & \underset{2}{U}^{i,\mu,\rho} & ,\cdots\cdots\cdots \\ \cdots\cdots\cdots, & U^{i,\mu,\rho} & ,\cdots\cdots\cdots \\ \cdots\cdots\cdots, & (\underset{2}{U}^{i,\mu,\rho} - \underset{1}{U}^{i,\mu,\rho}) & ,\cdots\cdots\cdots \end{bmatrix}$$

$$= [\cdots\cdots\cdots, W^{i,\mu,\rho}, \cdots\cdots\cdots] , \qquad (47)$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i,$$
$$i = 1,2,\ldots\ldots,\ell,$$

or in a compact form as

$$[G_P,G_I,G_D]\ U = W . \qquad (48)$$

The matrix U is square with dimension η, if U is invertible then

$$[G_P,G_I,G_D] = WU^{-1} . \qquad (49)$$

The determination of the feedback gain matrices G_P, G_I and G_D of the PID controller using (49) will complete the design procedure. In the previous analysis, it was assumed that all the eigenvalues not equal zero, however the case where all the eigenvalues are identically zero has practical application (deadbeat response). Therefore the case where an eigenvalue is zero will be discussed in the next section.

Eigenvalue λ_i Equal Zero

Let λ_i be zero eigenvalue, then (40-42) reduce to

$$\underset{2}{U}^{i,\mu,\rho} = \underset{1}{U}^{i,\mu,\rho-1}, \qquad (50)$$

$$-B G_D \underset{1}{U}^{i,\mu,\rho} + (A+B(G_P+G_D))\underset{2}{U}^{i,\mu,\rho} + B G_I U^{i,\mu,\rho} = \underset{2}{U}^{i,\mu,\rho-1}, \qquad (51)$$

$$Q_I \underset{2}{U}^{i,\mu,\rho} + U^{i,\mu,\rho} = U^{i,\mu,\rho-1} . \qquad (52)$$

From (50) and (52), we get

$$U^{i,\mu,\rho} = U^{i,\mu,\rho-1} - Q_I \underset{1}{U}^{i,\mu,\rho-1} . \qquad (53)$$

Examination of (50) and (53) indicates that $\underset{2}{U}^{i,\mu,\rho}$ and $U^{i,\mu,\rho}$ can be expressed in terms of $\underset{1}{U}^{i,\mu,\rho-1}$. Using (50) and substituting (53) into (51) we obtain

$$A\underset{1}{U}^{i,\mu,\rho-1} + B[G_P \underset{1}{U}^{i,\mu,\rho-1} + G_I(U^{i,\mu,\rho-1} - Q_I \underset{1}{U}^{i,\mu,\rho-1}) +$$
$$+ G_D(\underset{1}{U}^{i,\mu,\rho-1} - \underset{1}{U}^{i,\mu,\rho})] = \underset{1}{U}^{i,\mu,\rho-2} , \qquad (54)$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i,$$

or in a matrix form

$$[A,B]\begin{bmatrix} U_1^{i,\mu,\rho-1} \\ W^{i,\mu,\rho-1} \end{bmatrix} = U_1^{i,\mu,\rho-2}, \tag{55}$$

$$\rho = 2,3,\ldots\ldots,m_{i,\mu}+1, \qquad \mu = 1,2,\ldots\ldots,k_i,$$

where

$$W^{i,\mu,\rho-1} = G_P U_1^{i,\mu,\rho-1} + G_I(U^{i,\mu,\rho-1} - Q_I U_1^{i,\mu,\rho-1}) + G_D(U_1^{i,\mu,\rho-1} - U_1^{i,\mu,\rho}). \tag{56}$$

For $\rho = 1$, (55) reduces to

$$-G_D U_1^{i,\mu,1} = 0. \tag{57}$$

Using (55) and (56) the real feedback gain matrices can be expressed as

$$[G_P,G_I,G_D]\begin{bmatrix} \ldots, & 0, & U_1^{i,\mu,1}, & U_1^{i,\mu,2}, & \ldots \\ \ldots, & 0, & U^{i,\mu,1} - Q_I U_1^{i,\mu,1}, & U^{i,\mu,2} - Q_I U_1^{i,\mu,2}, & \ldots \\ \ldots, & -U_1^{i,\mu,1}, & U_1^{i,\mu,1} - U_1^{i,\mu,2}, & U_1^{i,\mu,2} - U_1^{i,\mu,3}, & \ldots \end{bmatrix}$$

$$= [\ldots., \quad 0, \quad W^{i,\mu,1}, \quad W^{i,\mu,2}, \ldots.] . \tag{58}$$

In the case where all the eigenvalues are identically zero, then from (57) and (58), it is clear that G_D is identically zero i.e. the resulting controller is a PI controller.

Eigenvalue $\lambda_i = 1$

Let λ_i be unity eigenvalue, then (40-42) reduce to

$$-U_1^{i,\mu,\rho} + U_2^{i,\mu,\rho} = U_1^{i,\mu,\rho-1}, \tag{59}$$

$$-BG_D U_1^{i,\mu,\rho} + ((A-I)+B(G_P+G_D))U_2^{i,\mu,\rho} + BG_I U^{i,\mu,\rho} = U_2^{i,\mu,\rho-1}, \tag{60}$$

$$Q_I U_2^{i,\mu,\rho} = U^{i,\mu,\rho-1} \tag{61}$$

We can assume without loss of generality that the matrix Q_I be expressed as $[I_q,0]$. Then from (61) it is obvious that q elements of $U_2^{i,\mu,\rho}$ are known in terms of $U^{i,\mu,\rho-1}$. The remaining (n-q) elements of $U_2^{i,\mu,\rho}$ and $U^{i,\mu,\rho}$ are to be assigned. From (59) we get

$$U_1^{i,\mu,\rho} = U_2^{i,\mu,\rho} - U_1^{i,\mu,\rho-1}, \tag{62}$$

which indicates that $U_1^{i,\mu,\rho}$ can be expressed in terms of $U_2^{i,\mu,\rho}$. Equation (60) can be written in a matrix form as

$$[A-I,B]\begin{bmatrix} U_2^{i,\mu,\rho} \\ W^{i,\mu,\rho} \end{bmatrix} = U_2^{i,\mu,\rho-1} \tag{63}$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i,$$

where

$$W^{i,\mu,\rho} = G_P U_2^{i,\mu,\rho} + G_I U^{i,\mu,\rho} + G_D U_1^{i,\mu,\rho-1} . \tag{64}$$

Using (63) and (64), we can express the real feedback gain matrices as

$$[G_P,G_I,G_D]\begin{bmatrix} \ldots\ldots, & U_2^{i,\mu,\rho} & ,\ldots\ldots \\ \ldots\ldots, & U^{i,\mu,\rho} & ,\ldots\ldots \\ \ldots\ldots, & U_1^{i,\mu,\rho-1} & ,\ldots\ldots \end{bmatrix} = [\ldots\ldots, W^{i,\mu,\rho} ,\ldots\ldots] . \tag{65}$$

For a general eigenvalue spectrum, we use the procedures discussed for the non-zero non-unity eigenvalue, zero eigenvalue and the present procedure for the unity eigenvalue, then employing (49), we can determine the PID controller gain matrices G_P, G_I and G_D.

ILLUSTRATIVE EXAMPLES

Two numerical examples will be considered to illustrate the proposed technique.

Example (1)

Consider a second order system of the form (27) where

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} , \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} ,$$

and two choices of Q_I will be treated:

(a) $Q_I = [0\ 1]$, (b) $Q_I = I_2$.

Using (35), it can be shown that the given system is state controllable.

The design objective is to determine the PID controller gain matrices such that the closed-loop system matrix A_c has the eigenvalue spectrum whose description:

$\lambda_1=0$, $k_1=1$, $m_1=1$, $\lambda_2=0.1$, $k_2=1$, $m_2=1$,
$\lambda_3=0.2$, $k_3=1$, $m_3=1$, $\lambda_4=-0.5$, $k_4=1$, $m_4=1$,

case (a) $\lambda_5=0.5$, $k_5=1$, $m_5=1$,
case (b) $\lambda_5=0.5$, $k_5=1$, $m_5=2$,

and the corresponding set of admissible generalized eigenvector subvectors $U_2^{i,\mu,\rho}$'s from :

$$U_1^{1,1,1}=\begin{bmatrix}1\\1\end{bmatrix},\quad U_2^{2,1,1}=\begin{bmatrix}-1\\1\end{bmatrix},\quad U_2^{3,1,1}=\begin{bmatrix}1\\-1\end{bmatrix},\quad U_2^{4,1,1}=\begin{bmatrix}-1\\2\end{bmatrix},$$

$$U_2^{5,1,1}=\begin{bmatrix}-1\\1\end{bmatrix},\quad U_2^{5,1,2}=\begin{bmatrix}2\\1\end{bmatrix}.$$

Note that

case (a) $q=1$, $i=1,2,3,4,5$, $m_{1,1}=m_{2,1}=m_{3,1}=m_{4,1}=m_{5,1}=1$
case (b) $q=2$, $i=1,2,3,4,5$, $m_{1,1}=m_{2,1}=m_{3,1}=m_{4,1}=1$, $m_{5,1}=2$.

Since the eigenvalue spectrum contains zero eigenvalue then the results obtained for the non zero eigenvalue and zero eigenvalue can be applied. Using (45), (47) and (57), the resulting feedback gain matrices of the PID controllers for cases (a), (b)

$$[G_P,G_I,G_D]=\begin{bmatrix}1.94 & 2.15 & 0.36 & 0.005 & -0.005\\ -1.16 & -0.16 & 0.00 & 0.00 & 0.00\end{bmatrix},$$

$$[G_P,G_I,G_D]=\begin{bmatrix}2.24 & 2.45 & 0.455 & 0.815 & 0.005 & -0.005\\ -1.33 & -0.33 & -0.250 & -0.250 & 0.000 & 0.000\end{bmatrix},$$

respectively. It can be shown that the resulting feedback gain matrices assign the desired eigenvalue spectrum and the set of admissible generalized eigenvectors to the closed-loop system matrix A_c. This will complete the design technique.

Example (2)

Consider a third order system of the form (27), where

$$A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\qquad B=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

and three choices of Q_I will be treated:

$$\text{(a) } Q_I=[1\ \ 0\ \ 0],\quad \text{(b) } Q_I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\quad \text{(c) } Q_I=I_3.$$

The objective is to design a PID controller such that the closed-loop system matrix A_c has the eigenvalue spectrum of the following description:

$\lambda_1 = 0, \quad k_1 = 3,$
case (a), $m_1 = 7,$
case (b), $m_1 = 8,$
case (c), $m_1 = 9,$

note $k_1=3$ the same as the dimension of the null space in (55) with $\rho=2$.

The corresponding set of admissible generalized eigenvector-subvectors $U_1^{i,\mu,\rho-1}$'s to be assigned is from:

$$U_1^{1,1,1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad U_1^{1,1,2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad U_1^{1,1,3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$U_1^{1,2,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad U_1^{1,2,2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad U_1^{1,2,3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$U_1^{1,3,1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad U_1^{1,3,2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad U_1^{1,3,3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Note that

case (a) $q=1$, $i=1$, $\mu=1,2,3$, $m_{1,1}=m_{1,2}=3$ and $m_{1,3}=1$,
case (b) $q=2$, $i=1$, $\mu=1,2,3$, $m_{1,1}=m_{1,2}=3$ and $m_{1,3}=2$,
case (c) $q=3$, $i=1$, $\mu=1,2,3$, $m_{1,1}=m_{1,2}=m_{1,3}=3$.

Since the eigenvalues are zeros, then the results obtained for zero eigenvalue can be applied. Using (55) and (56), the resulting feedback gain matrices of the PID controllers for cases (a), (b) and (c) are:

$$[G_P, G_I, G_D] = \begin{bmatrix} -2 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0.50 & -1.50 & 0.5 & 0 & 0 & 0 & 0 \\ -0.75 & 0.25 & -0.75 & -0.5 & 0 & 0 & 0 \end{bmatrix},$$

$$[G_P, G_I, G_D] = \begin{bmatrix} -2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 1 & -2 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$[G_P, G_I, G_D] = \begin{bmatrix} -2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

respectively. Substituting the numerical values of G_P, G_I and G_D and the system A,B matrices into the expression of A_c, it can be shown that the closed-loop system matrix A_c has zero eigenvalues and the plant state $Q_I X(k)$ will reach $Q_I X_d$ in two samples. Note for deadbeat response (i.e. assigned eigenvalues are identically zero), the resulting controller is a PI controller. This will complete the design procedure.

PART 3. PID CONTROLLER DESIGN UTILIZING OUTPUT FEEDBACK

PROBLEM STATEMENT

Consider the linear discrete-time, time-invariant multivariable system described in state space form as follows

$$\begin{aligned} X(k+1) &= AX(k) + BU(k) \\ Y(k) &= CX(k) \\ k &= 0,1,2,..... \end{aligned} \tag{66}$$

where $X(k)$ and $Y(k)$ are the state and output vectors and of dimensions n and ℓ respectively, at sample k, $U(k)$ is the input vector and of dimension m, A,B and C are constant matrices of appropriate dimensions and $\ell, m \leq n$. Let us introduce an m-dimensional integrator state vector $\mathcal{X}(k)$ which satisfies

$$\begin{aligned} \mathcal{X}(k+1) &= \mathcal{X}(k) + T(Q_I X(k) - Q_I X_d) \ , \\ k &= 0,1,2,..... \end{aligned} \tag{67}$$

where Q_I is a matrix of dimension mxn, $Q_I X_d$ is a desired vector for $Q_I X(k)$ and T is the sampling interval and can be set equal one. The state feedback law is assumed to be of the form

$$U(k) = G_P^* X(k) + G_I^* \mathcal{X}(k) + G_D^* X(k-1) \ , \tag{68}$$

where G_P^*, G_I^* and G_D^* are constant matrices representing the proportional, integral and derivative feedback gain matrices, of the PID controller and of dimension mxn. Substitution of (68) in (66) yields the closed-loop system

$$X(k+1) = (A+BG_P^*)X(k) - BG_D^* X(k-1) + BG_I^* \mathcal{X}(k). \tag{69}$$

Let us denote $X(k-1)$ and $X(k)$ by $Z_1(k)$ and $Z_2(k)$, respectively, then (66) and (67) can be written in state variable form as

$$\begin{aligned} Z_c(k+1) &= A_c Z_c(k) + \Theta, \\ Y(k) &= [0,C,0]Z_c(k) \end{aligned} \tag{70}$$

where

$$A_c = \begin{bmatrix} 0 & I_n & 0 \\ -BG_D^* & A+BG_P^* & BG_I^* \\ 0 & Q_I & I_m \end{bmatrix}, \qquad \Theta = \begin{bmatrix} 0 \\ 0 \\ -Q_I X_d \end{bmatrix}, \tag{71}$$

$$Z_c = \begin{bmatrix} Z_1(k) \\ Z_2(k) \\ X(k) \end{bmatrix} \tag{72}$$

and I_n is identity matrix of dimensions n. The closed-locp matrix A_c has a dimension 2n+m. The controllability condition established in PART 2. is valid in this case and will not be repeated.

The design problem considered in this part can be presented as two-stage design procedure:

Stage 1. State Feedback Design

Determine the real matrices G_P^*, G_I^* and G_D^* such that the closed-loop system matrix A_c (71), of the system represented by (70) via a feedback law of the form (68) and the integrator is governed by (67), has a prescribed set of self-conjugate eigenvalues and a corresponding set of admissible self-conjugate generalized eigenvectors.

Step 2. Output Feedback Implementation

Let U(k) has the alternative structure:

$$U(k) = G_P L_P Y(k) + G_I Y(k) + G_D L_D (Y(k)-Y(k-1)) \tag{73}$$

where

$$Y(k+1) = Y(k) + (L_I Y(k) - L_I Y_d) \tag{74}$$

where the matrices L_p, L_I and L_D are of dimensions ($\ell_p \times \ell$), ($\ell_i \times \ell$) and ($\ell_d \times \ell$) respectively, and ℓ_p, ℓ_i, ℓ_d satisfy:

$$\begin{aligned} \ell_p+\ell_i+\ell_d &\leq 2n+m \\ \ell_p+\ell_i &\leq n+m \\ \ell_p, \ell_i, \ell_d &\leq \ell \end{aligned} \tag{75}$$

and the matrices L_pC, L_IC and L_DC are of full rank. The system designer has the freedom in selecting the matrices L_p, L_I and L_D.

Then we determine the real output feedback matrices G_p, G_I and G_D in terms of the state feedback matrices G_p^*, G_I^* and G_D^*. Note that the closed-loop

system matrix in case of feedback law of the form (73) is given by

$$A_{co} = \begin{bmatrix} 0 & I_n & 0 \\ -BG_DL_DC & A+BG_PL_PC & BG_I \\ & +BG_DL_DC & \\ 0 & L_IC & I_{\ell_i} \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 \\ 0 \\ -L_IY_d \end{bmatrix} \tag{76}$$

where I_{ℓ_i} and I_n are identity matrices of dimensions ℓ_i and n, respectively. The design problem presented can be viewed as state-design and output-implementation.

STAGE 1. STATE FEEDBACK EIGENVALUE-GENERALIZED EIGENVECTOR ASSIGNMENT

Eigenvalue Not Equal Zero Or One

Following similar argument as presented in PART 2., we get

$$[A-\lambda_i I, B] \begin{bmatrix} U_2^{i,\mu,\rho} \\ W^{i,\mu,\rho} \end{bmatrix} = U_2^{i,\mu,\rho-1}, \tag{77}$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i,$$
$$i = 1,2,\ldots\ldots,\ell,$$

where

$$W^{i,\mu,\rho} = G_P^* U_2^{i,\mu,\rho} + G_I^* U^{i,\mu,\rho} + G_D(-U_1^{i,\mu,\rho}). \tag{78}$$

The subvectors $U_1^{i,\mu,\rho}$ and $U^{i,\mu,\rho}$ are expressed in terms of $U_2^{i,\mu,\rho}$ as in (43) and (44). Using (77) and (78) the state feedback gain matrices G_P^*, G_I^* and G_D^* are expressed as

$$[G_P^*, G_I^*, G_D^*] \begin{bmatrix} \ldots\ldots\ldots, & U_2^{i,\mu,\rho}, & \ldots\ldots\ldots \\ \ldots\ldots\ldots, & U^{i,\mu,\rho}, & \ldots\ldots\ldots \\ \ldots\ldots\ldots, & -U_1^{i,\mu,\rho}, & \ldots\ldots\ldots \end{bmatrix}$$
$$= [\ldots\ldots\ldots, W^{i,\mu,\rho}, \ldots\ldots\ldots], \tag{79}$$

$$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots\ldots,k_i,$$
$$i = 1,2,\ldots\ldots,\ell,$$

or in a compact form as

$$[G_P^*, G_I^*, G_D^*]\, U = W. \tag{80}$$

The matrix U is square with dimension 2n+m, if U is invertible then

$$[G_P^*, G_I^*, G_D^*] = WU^{-1}. \tag{81}$$

The determination of the feedback gain matrices G_P^*, G_I^* and G_D^* of the PID controller using (81) will complete Stage 1. design procedure. In the previous analysis, it was assumed that all the eigenvalues not equal zero, however the case where all the eigenvalues are identically zero has practical application (deadbeat response). Therefore the case where an eigenvalue is zero will be discussed in the next section.

Eigenvalue λ_i Equal Zero

Let λ_i be zero eigenvalue and following the same argument as in PART 2. we get

$$[A,B]\begin{bmatrix} U_1^{i,\mu,\rho-1} \\ W^{i,\mu,\rho-1} \end{bmatrix} = U_1^{i,\mu,\rho-2}, \tag{82}$$

$$\rho = 2,3,\ldots\ldots,m_{i,\mu}+1, \qquad \mu = 1,2,\ldots\ldots\ldots,k_i,$$

where

$$W^{i,\mu,\rho-1} = G_P^* U_1^{i,\mu,\rho-1} + G_I^*(U^{i,\mu,\rho-1} - Q_I U_1^{i,\mu,\rho-1}) + G_D^*(-U_1^{i,\mu,\rho}). \tag{83}$$

The subvectors $U_1^{i,\mu,\rho}$ and $U^{i,\mu,\rho}$ are expressed in terms of $U_1^{i,\mu,\rho-1}$ as in (50) and (53). Using (82) and (83) the state feedback gain matrices G_P^*, G_I^* and G_D^* can be expressed as

$$[G_P^*, G_I^*, G_D^*]\begin{bmatrix} \ldots, & 0 & , U_1^{i,\mu,1} & , U_1^{i,\mu,2} & ,\ldots \\ \ldots, & 0 & , U^{i,\mu,1} - Q_I U_1^{i,\mu,1} & , U^{i,\mu,2} - Q_I U_1^{i,\mu,2} & ,\ldots \\ \ldots, & -U_1^{i,\mu,1} & , -U_1^{i,\mu,2} & , -U_1^{i,\mu,3} & ,\ldots \end{bmatrix} = [\ldots, \; 0 \; , W^{i,\mu,1}, W^{i,\mu,2}, \ldots] \tag{84}$$

In the case where all the eigenvalues are identically zero, then from (84), it is clear that G_D is identically zero i.e. the resulting controller is a PI controller.

Eigenvalue λ_i=1

Let λ_i be unity eigenvalue and following the same argument as in PART 2., we get

$$[A-I,B]\begin{bmatrix} U_2^{i,\mu,\rho} \\ W^{i,\mu,\rho} \end{bmatrix} = U_2^{i,\mu,\rho-1} \tag{85}$$

$\rho = 1,2,\ldots\ldots,m_{i,\mu}, \qquad \mu = 1,2,\ldots\ldots,k_i,$

where

$$W^{i,\mu,\rho} = G_P^* U_2^{i,\mu,\rho} + G_I^* U^{i,\mu,\rho} + G_D^* U_1^{i,\mu,\rho} \; . \tag{86}$$

The subvector $U_1^{i,\mu,\rho}$ is expressed in terms of $U_2^{i,\mu,\rho}$ as in (62), and $U^{i,\mu,\rho}$ to be assigned. Using (85) and (86), we can express the real feedback gain matrices as

$$[G_P^*, G_I^*, G_D^*]\begin{bmatrix} \ldots\ldots, & U_2^{i,\mu,\rho}, \ldots\ldots \\ \ldots\ldots, & U^{i,\mu,\rho}, \ldots\ldots \\ \ldots\ldots, & U_1^{i,\mu,\rho}, \ldots\ldots \end{bmatrix} = [\ldots\ldots, W^{i,\mu,\rho}, \ldots\ldots] \tag{87}$$

For a general eigenvalue spectrum, we use the procedures discussed for the non-zero non-unity eigenvalue, zero eigenvalue and the present procedure for the unity eigenvalue, then employing (81), we can determine the state feedback gain matrices G_P^*, G_I^* and G_D^* for Stage 1. State Feedback Design.

STAGE 2. OUTPUT FEEDBACK PID CONTROLLER DESIGN

In Stage 1. Design, the structure of the matrix Q_I is constrained by the controllability requirement of the augmented system (71), and the system designer has a control on the choice of the matrices L_P, L_I and L_D. From (68) and (73) the following relations can be derived.

$$G_I = G_I^* Q_I (L_I C)'((L_I C)(L_I C)')^{-1}, \tag{88}$$

$$G_D = G_D^* (L_D C)'((L_D C)(L_D C)')^{-1}, \tag{89}$$

$$G_P = (G_P^* - G_D^*)(L_P C)'((L_P C)(L_P C)')^{-1}. \tag{90}$$

The determination of the output feedback matrices G_P, G_I and G_D will complete Stage 2 procedure and the output PID controller design.

ILLUSTRATIVE EXAMPLE

Consider the same system considered in PART 2., Example 2 with

output matrix

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

case (a) $L_P = [1\ 1]$, $L_I = [1\ 1]$, $L_D = [1\ 0]$,

case (b) $L_P = I_2$, $L_I = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $L_D = I_2$,

The objective is to determine the output feedback gain matrices G_P, G_I and G_D of the feedback law of the form (73) which drives the output $L_I Y(k)$ to a desired value $L_I Y_d$ in a minimum time.

Stage 1. Procedure

Determine the feedback gain matrices G_P^*, G_I^* and G_D^* which assign the eigenvalue spectrum and the corresponding generalized eigenvectors (given in PART 2., Example 2 with $Q_I = I_3$) to the closed-loop matrix A_c (71). Note the resulting feedback gain matrices will drive any initial state $Z_c(0)$ (70) to the desired vector in two samples. By applying results obtained in PART 3. we get

$$G_P^*, G_I^*, G_D^* = \begin{bmatrix} -2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Starting from zero initial state, it can be shown that the state of the plant X(k) will reach the desired state [2,2,2] in two samples. This will complete Stage 1. Procedure.

Stage 2. Procedure

Determine the output feedback gain matrices G_P, G_I and G_D of the digital PID controller in terms of G_P^*, G_I^* and G_D^*. Applying the relations (88-90) for cases (a) and (b) we get

case (a) $$[G_P, G_I, G_D] = \begin{bmatrix} -1 & -1/3 & 0 \\ 0 & 0 & 0 \\ -2/3 & -1/3 & 0 \end{bmatrix},$$

case (b) $$[G_P, G_I, G_D] = \begin{bmatrix} 0 & -1.5 & -0.5 & 0 & 0 & 0 \\ -2 & 1 & 1.5 & -1 & 0 & 0 \\ 0 & -1 & -0.5 & 0 & 0 & 0 \end{bmatrix}.$$

Let the desired values for the system $L_I Y(k)$ are:

case (a) $L_I Y_d = [x_1(k) + x_2(k) + x_3(k)] \rightarrow [6]$

case (b) $L_I Y_d = \begin{bmatrix} x_1(k) + x_3(k) \\ x_1(k) + x_2(k) + x_3(k) \end{bmatrix} \rightarrow \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

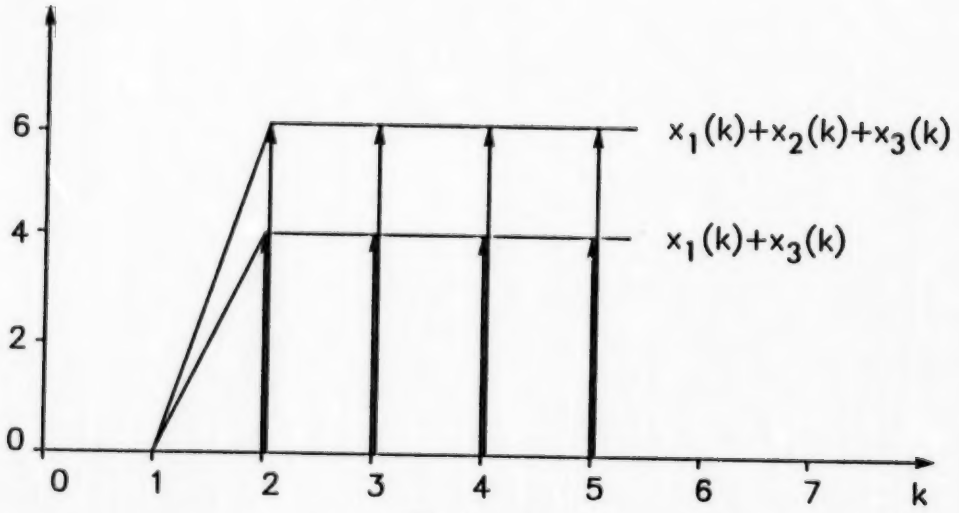

Figure 3
System Output Response Using Deadbeat PID Controller

Then it can be verified that the system $L_I Y(k)$ response as shown in Figure 3, and will reach the desired values in two samples. This will complete Stage 2. Procedure.

CONCLUSIONS

In this contribution, time-domain design techniques for digital PI and PID controllers for linear multivariable discrete-time system have been presented. Two situations are considered where the system state or output is available for measurement. In the case of state PI and PID controller, the resulting gain matrices assign a prescribed set of self-conjugate eigenvalues and a corresponding set of admissible self-conjugate generalized eigenvectors to the closed-loop system matrix. However, in the case of output PID controller a two-stage design approach is adopted, where in stage 1 we determine state feedback gain to assign a prescribed set of eigenvalue-generalized eigenvector to a particular closed-loop matrix, then in stage 2 we utilize the state feedback gain matrices to determine the output feedback matrices of the output PID controller. It should be noticed that the procedure utilized in the design of output PID controller is similar to the one considered by (Khalifa and Hanafy. (1984)), except it is adopted to handle eigenvalue-eigenvector assignment. The case of designing output PI controller can be handled in a similar fashion as the design of output PID controller.

The proposed techniques appear to be flexible, straightforward and computationally simpler. Moreover, they do not impose any structural constraints on the controller feedback gain matrices of being unit rank, and the computation of the open-loop transfer function matrix is avoided. It has been found that in the case where the assigned eigenvalue spectrum is zero, the derivative feedback gain matrix is identically zero i.e. the resulting controller is a PI controller. The freedom in selecting the generalized eigenvectors could be utilized in minimizing the eigenvalues sensitivities, quadratic performance index or any other performance criterion.

The eigenstructure approach adopted in this paper is similar to the eigenstructure approach utilized in the eigenvalue-generalized eigenvector assignment using state feedback (Klein and Moore (1977) and Porter and D'Azzo (1978)).

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